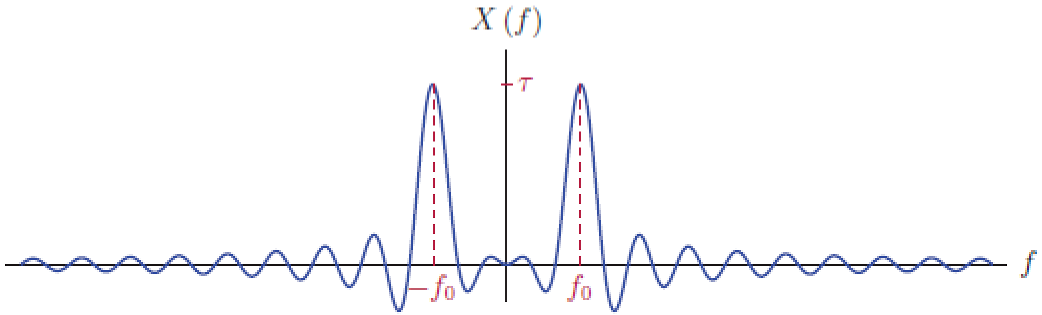
CECC507: Signals and Systems

Lecture Notes 3: Analyzing Continuous Time Systems in the Time Domain



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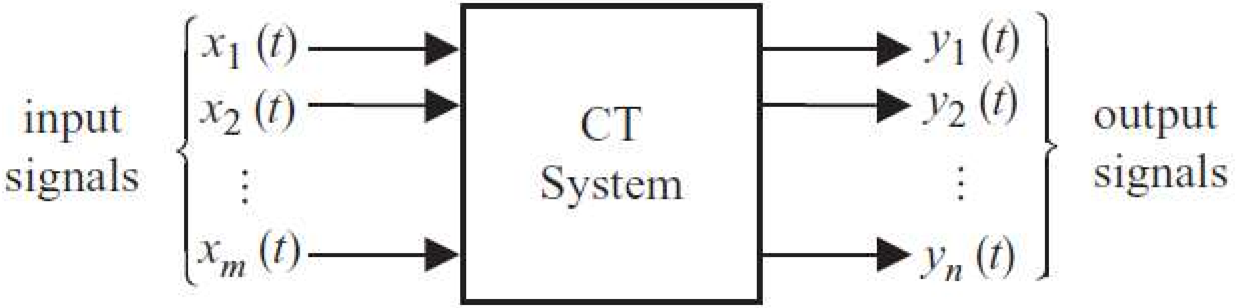
Chapter 2

Analyzing Continuous Time Systems in the Time Domain

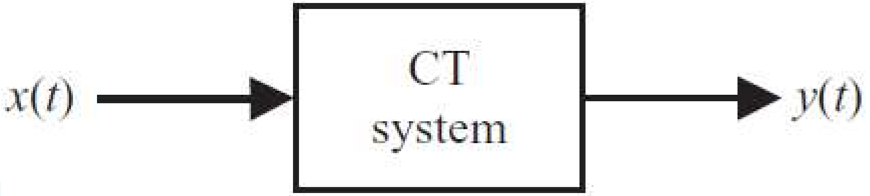
1. Linearity and Time Invariance
2. Differential Equations for Continuous-Time Systems
3. Constant-Coefficient Ordinary Differential Equations
4. Block Diagram Representation of Continuous-Time Systems
5. Impulse Response and Convolution
6. Causality and Stability in Continuous-Time Systems

Introduction

- In general, a system is any physical entity that takes in a set of one or more physical signals and, in response, produces a new set of one or more physical signals.

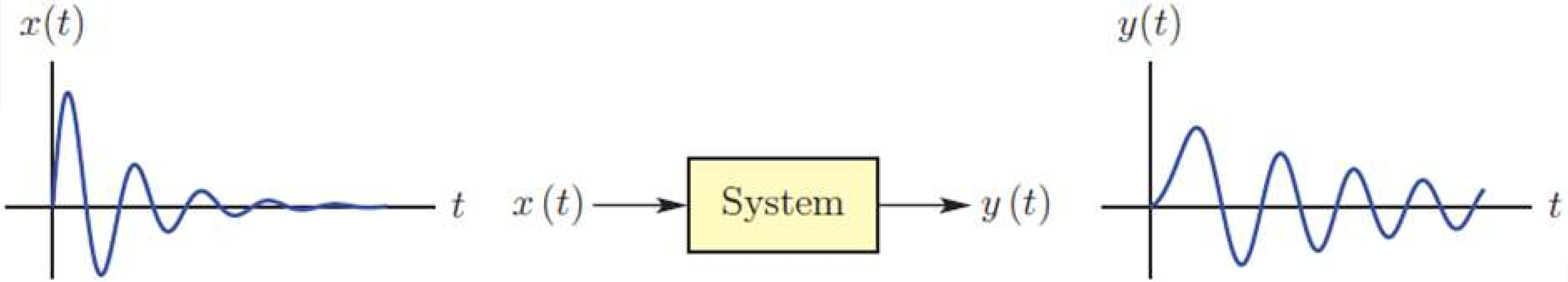


Multiple-input, multiple-output (MIMO) CT system



Single-input, single-output CT system

- If we focus our attention on single-input/single-output systems, the interplay between the system and its input and output signals can be graphically illustrated as:



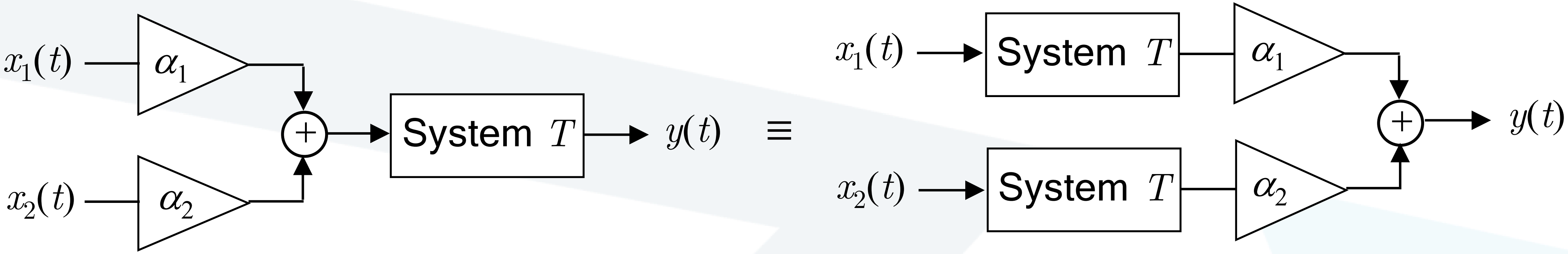
- The input signal is $x(t)$, and the output signal is $y(t)$. The system may be denoted by the equation $y(t) = T\{x(t)\}$, where $T\{.\}$ indicates a transformation.

1. Linearity and Time Invariance

Linearity in continuous-time systems

- A system T is linear, if for all functions x_1 and x_2 and all constants α_1 and α_2, the following condition holds:

$$T\{\alpha_1 x_1(t) + \alpha_2 x_2(t)\} = \alpha_1 T\{x_1(t)\} + \alpha_2 T\{x_2(t)\}.$$



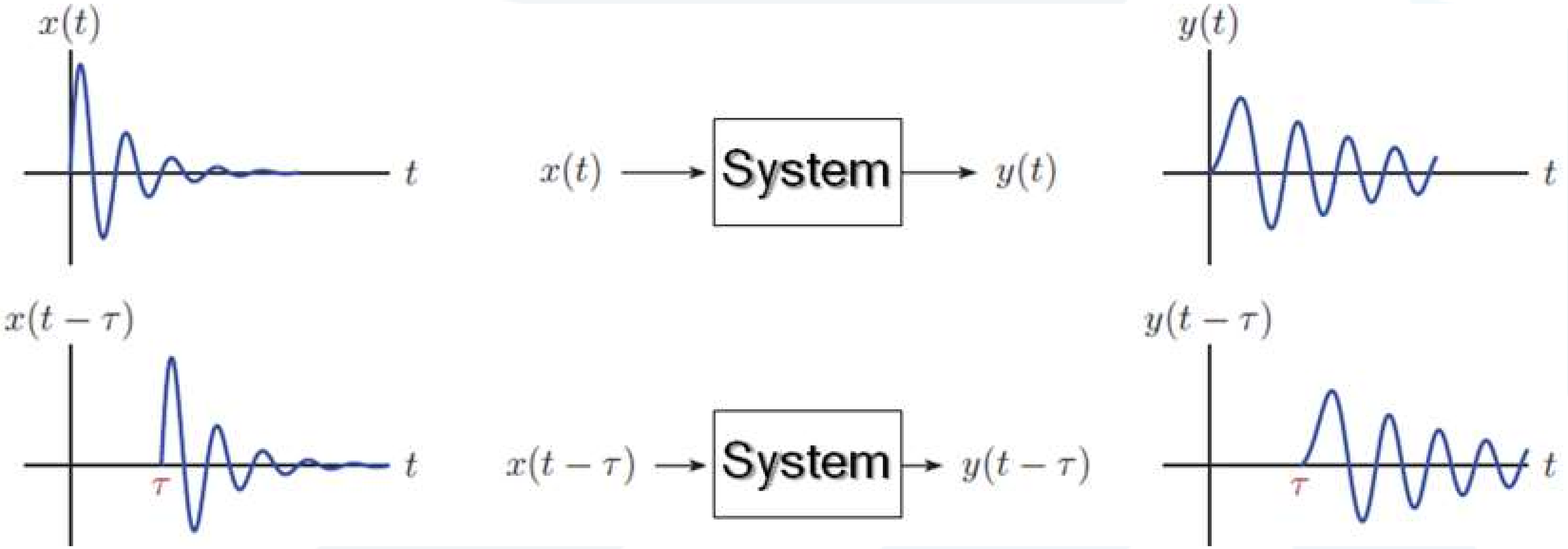
- The linearity property is also referred to as the superposition property.
- Linear systems are much easier to design and analyze than nonlinear systems.
- Example 1: For each, determine if the system is linear or not:

a. $y(t) = 5x(t)$ √ b. $y(t) = 5x(t) + 3$ ✗

c. $y(t) = 3[x(t)]^2$ ✗ d. $y(t) = \cos(x(t))$ ✗

Time Invariance in continuous-time systems

- A system T is said to be time invariant (TI) if, for every function x and every real constant τ, the following condition holds: $T\{x(t)\} = y(t) \Rightarrow T\{x(t-\tau)\} = y(t-\tau)$.

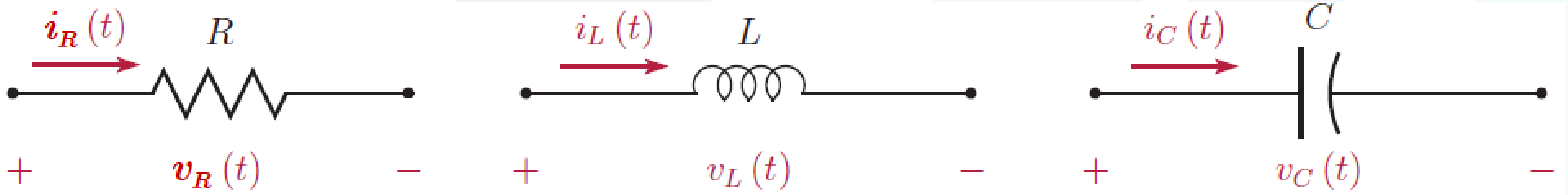


- Example 2: For each, determine whether the system is time-invariant or not:

a. $y(t) = 5x(t)$ √ b. $y(t) = 3\cos(x(t))$ √ c. $y(t) = 3\cos(t)x(t)$ ✗

2. Differential Equations for Continuous-Time Systems

- One method of representing the relationship established by a system between its input and output signals is a differential equation.
- model for an ideal resistor is: $v_R(t) = R i_R(t)$
- model for an ideal inductor is: $v_L(t) = L \frac{di_L(t)}{dt}$
- model for an ideal capacitor is: $i_C(t) = C \frac{dv_C(t)}{dt}$

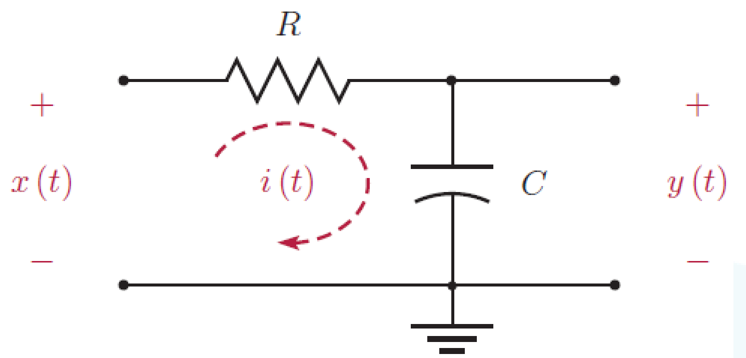


- Example 3: Differential equation for simple RC circuit

$$v_R(t) = Ri(t), \qquad i(t) = C\frac{dy(t)}{dt}$$

$$RC\frac{dy(t)}{dt} + y(t) = x(t)$$

$$\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}x(t)$$

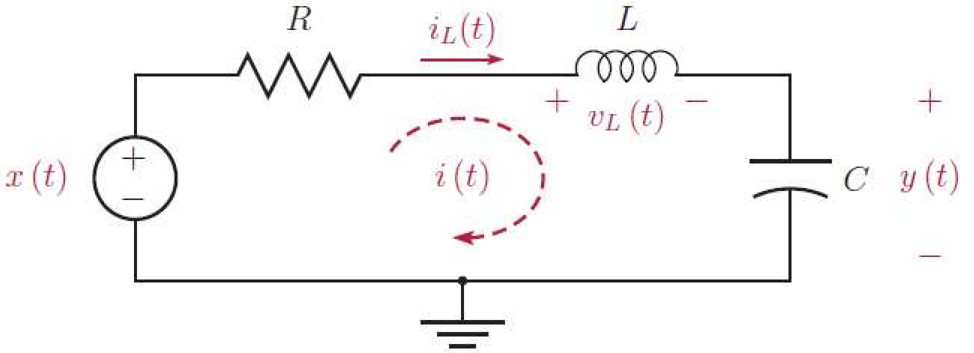


- Example 4: Differential equation for RLC circuit

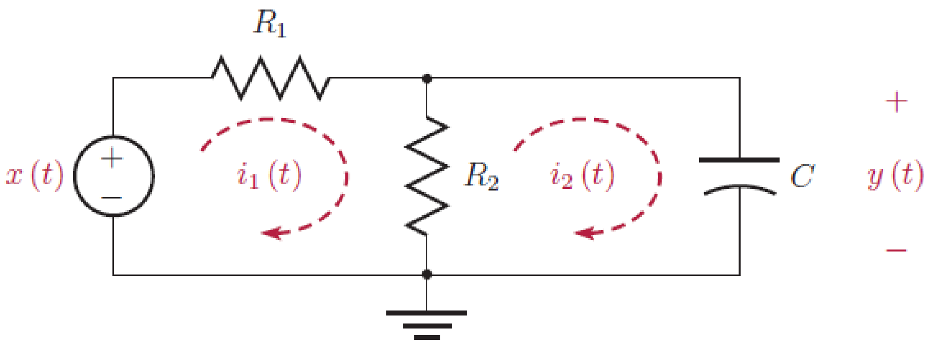
$$v_L(t) = L\frac{di(t)}{dt}, \quad i(t) = C\frac{dy(t)}{dt}$$

$$-x(t) + Ri(t) + v_L(t) + y(t) = 0$$

$$\frac{d^2y(t)}{dt^2} + \frac{R}{L}\frac{dy(t)}{dt} + \frac{1}{LC}y(t) = \frac{1}{LC}x(t)$$



- Example 5: Another RC circuit



$$-x(t) + R_1 i_1(t) + R_2[i_1(t) - i_2(t)] = 0$$
$$R_2[i_2(t) - i_1(t)] + y(t) = 0$$
$$i_2(t) = C\frac{dy(t)}{dt} \Rightarrow i_1(t) = C\frac{dy(t)}{dt} + \frac{1}{R_2}y(t)$$
$$-x(t) + R_1 C\frac{dy(t)}{dt} - \frac{R_1 + R_2}{R_2}y(t) = 0 \Rightarrow \frac{dy(t)}{dt} + \frac{R_1 + R_2}{R_1 R_2 C}y(t) = \frac{1}{R_1 C}x(t)$$

3. Constant-Coefficient Ordinary Differential Equations

- In general, CTLTI systems can be modeled with ODEs that have constant coefficients.

$$a_N\frac{d^N y(t)}{dt^N} + a_{N-1}\frac{d^{N-1}y(t)}{dt^{N-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t)$$
$$= b_M\frac{d^M x(t)}{dt^N} + b_{M-1}\frac{d^{M-1}x(t)}{dt^{M-1}} + \cdots + b_1\frac{dx(t)}{dt} + b_0 x(t)$$

or it can be expressed in the form: $\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$

- In general, a constant-coefficient ODE has a family of solutions. In order to find a unique solution for $y(t)$, initial values of the output signal and its first $N-1$ derivatives need to be specified at a time instant $t = t_0$. We need to know:

$$y(t_0), \quad \left.\frac{dy(t)}{dt}\right|_{t=t_0}, \cdots, \quad \left.\frac{d^{N-1}y(t)}{dt^{N-1}}\right|_{t=t_0}$$

to find the solution for $t > t_0$

- The DE $\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$ represents an LTI system provided that:

$$y(t_0) = 0, \quad \left.\frac{dy(t)}{dt}\right|_{t=t_0} = 0, \cdots, \quad \left.\frac{d^{N-1}y(t)}{dt^{N-1}}\right|_{t=t_0} = 0$$

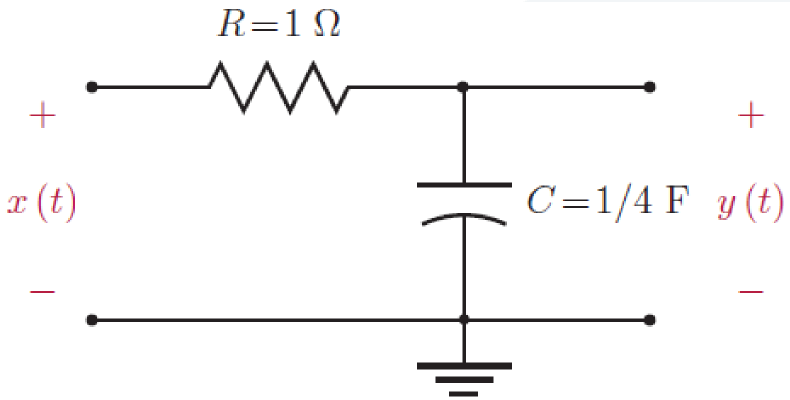
Solving Differential Equations

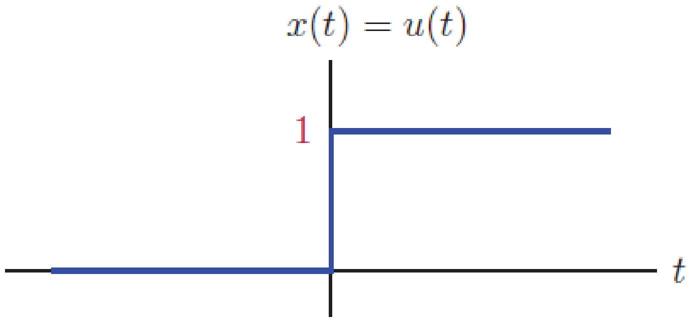
Solution of the first-order differential equation

- The differential equation $\dfrac{dy(t)}{dt} + \alpha y(t) = r(t), \quad y(t_0)$: specified

  is solved as $y(t) = e^{-\alpha(t-t_0)}y(t_0) + \int_{t_0}^{t} e^{-\alpha(t-\tau)}r(\tau)d\tau$

- Example 5: Unit-step response of the simple RC circuit ($y(0) = 0$)



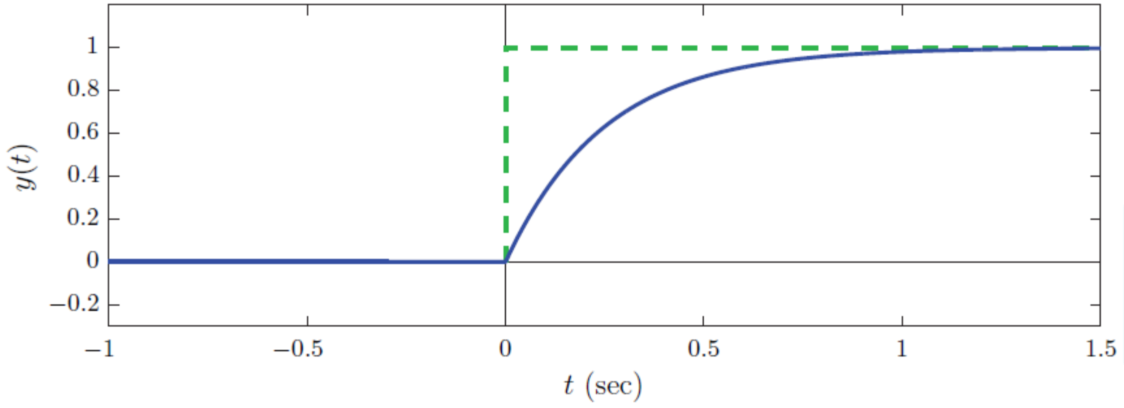


The DE of the circuit is: $\frac{dy(t)}{dt} + \frac{1}{RC} y(t) = \frac{1}{RC} u(t) \Rightarrow \frac{dy(t)}{dt} + 4y(t) = 4u(t)$

$$y(t) = \int_0^t e^{-(t-\tau)/RC} \frac{1}{RC} u(\tau) d\tau$$

$$= \frac{e^{-t/RC}}{RC} \int_0^t e^{\tau/RC} d\tau = 1 - e^{-t/RC}, \quad t \geq 0$$

$$y(t) = (1 - e^{-t/RC})u(t) = (1 - e^{-4t})u(t)$$

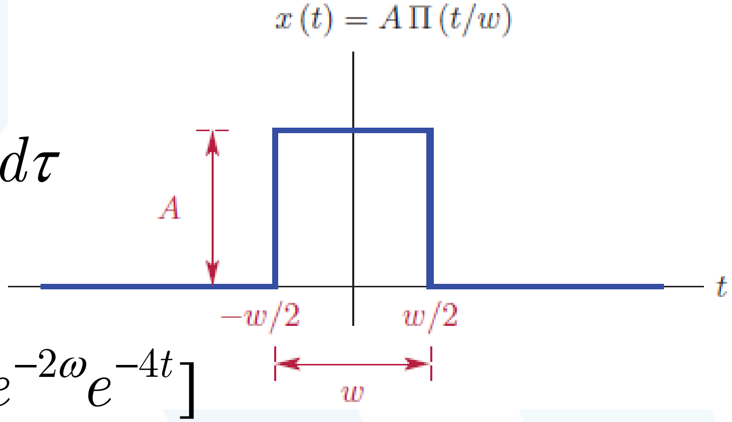


- Example 6: Pulse response of the simple RC circuit

$$\frac{dy(t)}{dt} + 4y(t) = 4\prod(t/\omega) \Rightarrow y(t) = \int_{-\omega/2}^{t} e^{-4(t-\tau)} 4A\prod(\tau/\omega) d\tau$$

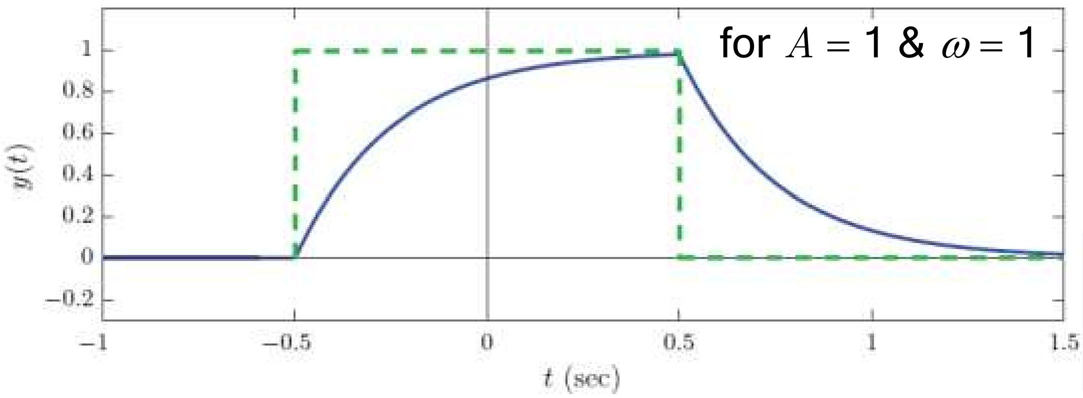
Case 1: $t \leq -\omega/2,\ y(t) = 0$

Case 2: $-\omega/2 < t \leq \omega/2, y(t) = 4A\int_{-\omega/2}^{t} e^{-4(t-\tau)} d\tau = A[1 - e^{-2\omega} e^{-4t}]$



Case 3: $t > \omega/2$, $y(t) = 4A\int_{-\omega/2}^{\omega/2} e^{-4(t-\tau)} d\tau = Ae^{-4t}[e^{2\omega} - e^{-2\omega}]$

$$y(t) = \begin{cases} 0, & t < -\frac{\omega}{2} \\ A[1 - e^{-2\omega}e^{-4t}], & -\frac{\omega}{2} < t \le \frac{\omega}{2} \\ Ae^{-4t}[e^{2\omega} - e^{-2\omega}], & t > \frac{\omega}{2} \end{cases}$$



Solution of the general differential equation

- To solve the general constant-coefficient DE we will consider two separate components of the output signal $y(t)$ as follows: $y(t) = y_h(t) + y_p(t)$.
- A particular solution is usually obtained by assuming an output of the same general form as the input.

Input signal	Particular solution
t^n	$k_n t^n + k_{n-1} t^{n-1} + ... k_1 t + k_0$ (Constant input is a special case with $n=0$)
$e^{\alpha t}$	$ke^{\alpha t}$, α is not the characteristic value (c.v.) $k_1 te^{\alpha t} + k_0 e^{\alpha t}$, α is the characteristic value with order 1 $k_k t^k e^{\alpha t} + k_{k-1} t^{k-1} e^{\alpha t} + ... k_1 te^{\alpha t} + k_0 e^{\alpha t}$, α is the c.v. with order k
$\cos(\omega t)$ or $\sin(\omega t)$	$k_1\cos(\omega t) + k_2\sin(\omega t)$

- $y_h(t)$, is the solution of the homogeneous DE found by ignoring the input signal, i.e. setting $x(t)$ and all of its derivatives equal to zero:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0$$

- $y_h(t)$ is called the natural response of the system.

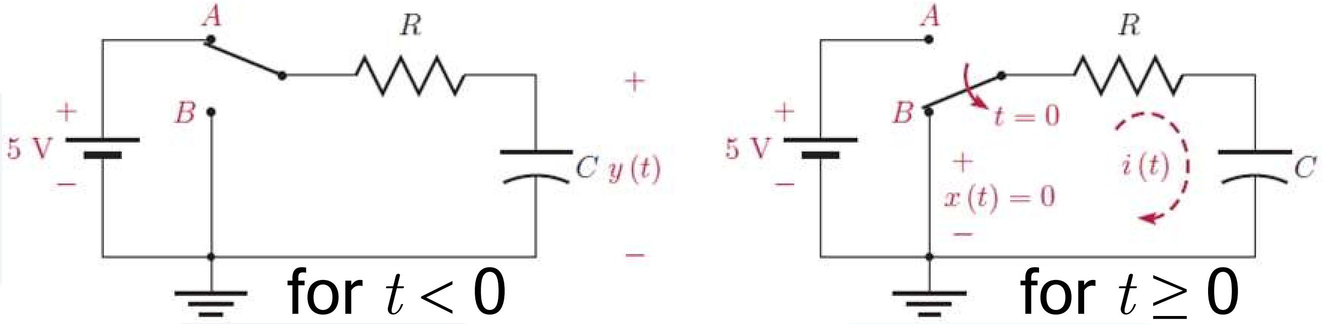
- For a stable system, $y_h(t)$ tends to gradually disappear in time. Because of this, it is also referred to as the transient response of the system.
- The second term $y_p(t)$ is due to the input signal $x(t)$ being applied to the system. It is referred to as the particular (forced) solution of the DE.
- $y_p(t)$ will be linked to the steady-state response of the system, that is, the response to an input signal that has been applied for a long enough time for the transient terms to die out.

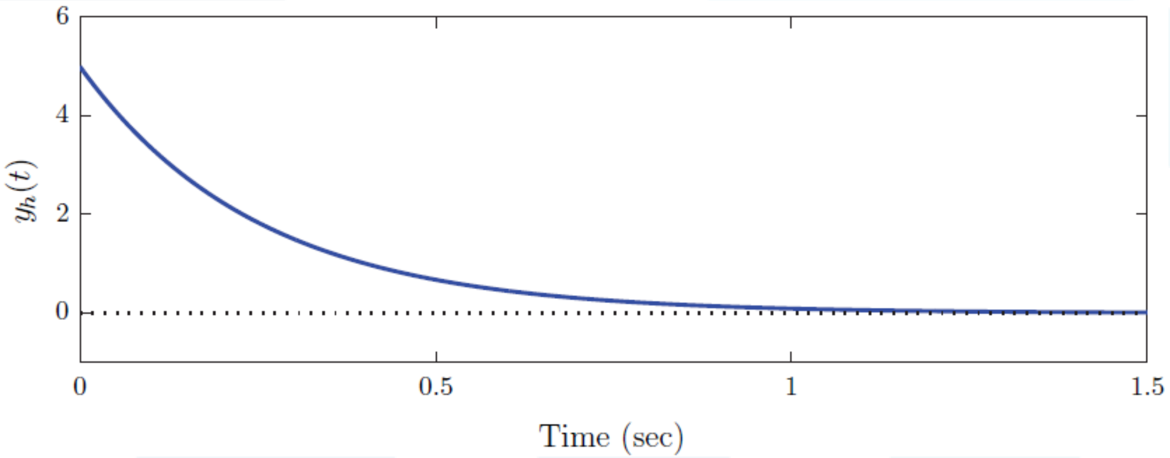
Finding the natural response of a continuous-time system

- Example 7: Natural response of the simple RC circuit

  Consider the RC circuit with $R = 1$ Ω and $C = 1/4$ F. Let the input terminals of the circuit be connected to a battery that supplies the circuit with an input voltage of 5 V up to the time instant $t = 0$.

$$\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = 0$$
$$\frac{dy(t)}{dt} + 4y(t) = 0,$$
$$y_h(t) = ce^{-4t},\ t \geq 0$$
$$y_h(0) = 5 \Rightarrow c = 5$$
$$y_h(t) = 5e^{-4t}u(t)$$





- **Example 8:** Natural response of a second-order system (RLC circuit)
  At time $t = 0$, the initial inductor current is $i(0) = 0.5$ A and the initial capacitor voltage is $y(0) = 2$ V. $x(t) = 0$. Determine the output voltage $y(t)$ if:

a. the element values are $R = 2\ \Omega$, $L = 1$ H and C = 1/26 F,

b. the element values are $R = 6\ \Omega$, $L = 1$ H and C = 1/9 F.

$$a.\ \frac{d^2y(t)}{dt^2} + \frac{R}{L}\frac{dy(t)}{dt} + \frac{1}{LC}y(t) = 0 \Rightarrow \frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 26y(t) = 0$$

$$y_h(t) = c_1e^{-t}\cos(5t) + c_2e^{-t}\sin(5t),\ t \geq 0$$

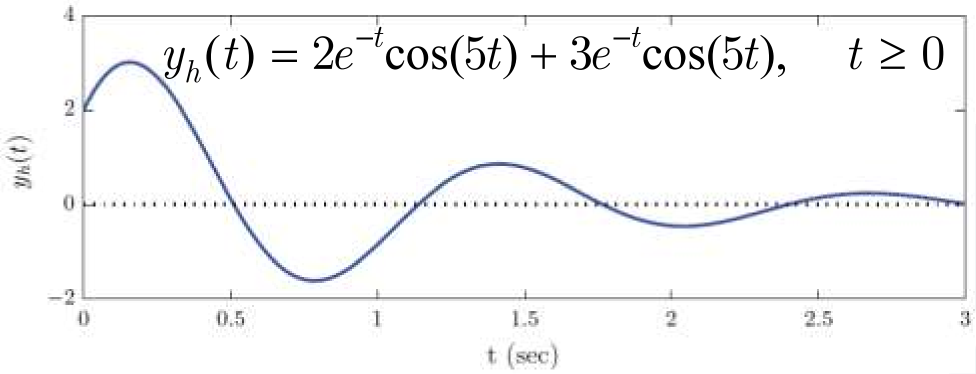
$$y_h(0) = 2, \quad i(0) = C\frac{dy_h}{dt}(0) = 0.5 \Rightarrow c_1 = 2, \quad c_2 = 3$$

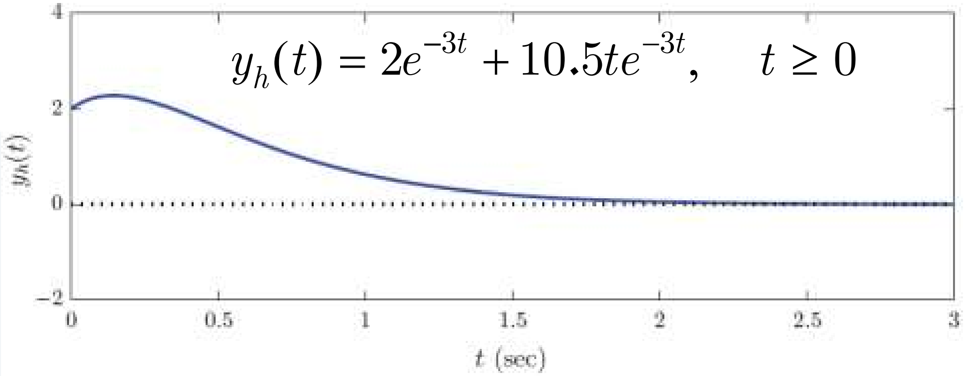
$$y_h(t) = (2e^{-t}\cos(5t) + 3e^{-t}\sin(5t))u(t)$$

$$b.\ \frac{d^2y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 9y(t) = 0 \Rightarrow y_h(t) = c_1e^{-3t} + c_2te^{-3t},\ t \geq 0$$

$$y_h(0) = 2, \quad i(0) = C\frac{dy_h}{dt}(0) = 0.5 \Rightarrow c_1 = 2, \quad c_2 = 10.5$$

$$y_h(t) = (2e^{-3t} + 10.5te^{-3t})u(t)$$





Finding the forced response of a continuous-time system

- **Example 9:** Forced response of the first-order system for sinusoidal input

  The initial value of the output signal is $y(0) = 5$. Determine the output signal in response to a sinusoidal input signal in the form $x(t) = 5\cos(8t)$.

$$\frac{dy(t)}{dt} + 4y(t) = 4x(t) \qquad y_h(t) = ce^{-4t},\ t \geq 0$$

$$y_p(t) = a\cos(8t) + b\sin(8t) \Rightarrow \frac{dy_p(t)}{dt} = -8a\sin(8t) + 8b\cos(8t)$$

$$-8a\sin(8t) + 8b\cos(8t) + 4a\cos(8t) + 4b\sin(8t) = 20\cos(8t) \Rightarrow a = 1,\ b = 2$$

$$y(t) = ce^{-4t} + \cos(8t) + 2\sin(8t),\ t \geq 0$$

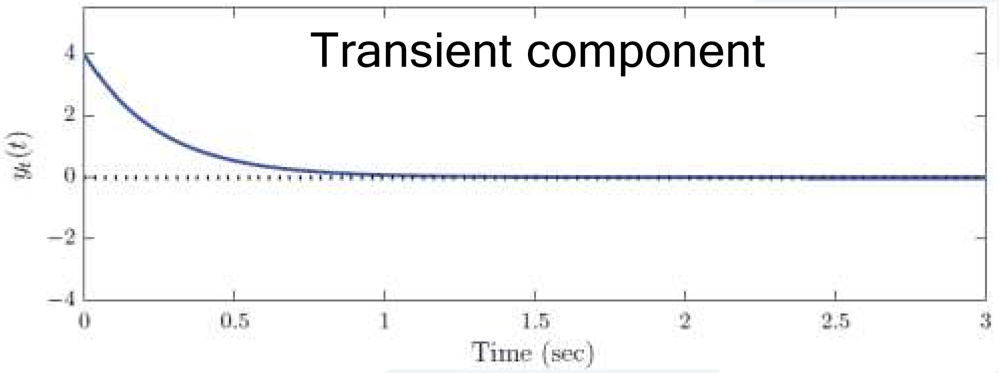
$$y(0) = 5 \Rightarrow c = 4 \Rightarrow y(t) = \underbrace{4e^{-4t}}_{y_t(t)} + \underbrace{\cos(8t) + 2\sin(8t)}_{y_{ss}(t)},\ t \geq 0$$

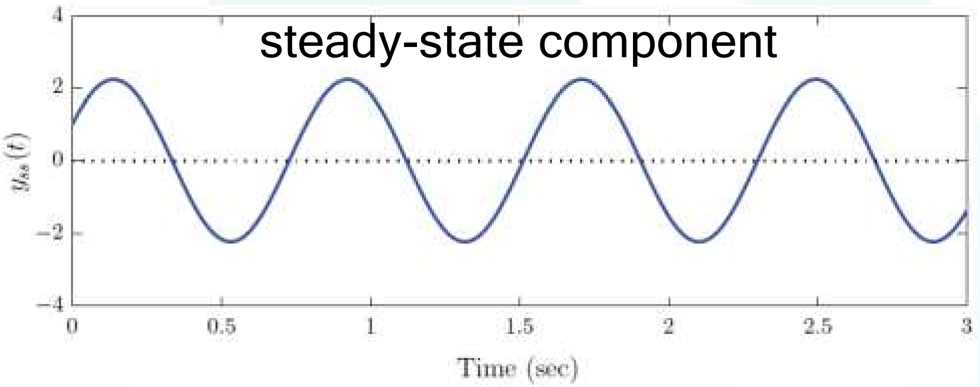
$$y_t(t) = 4e^{-4t},\ \lim_{t\to\infty}\{y_t(t)\} = 0$$

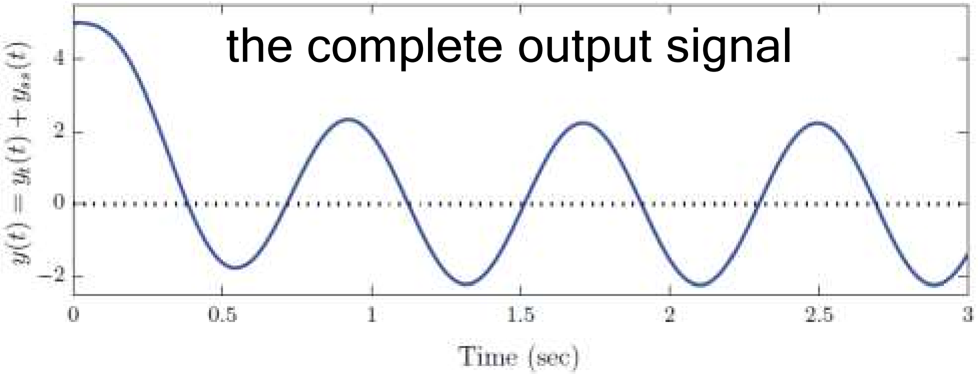
$y_t(t)$: transient response of the system

$$y_{ss}(t) = \cos(8t) + 2\sin(8t)$$

$y_{ss}(t)$: steady-state response of the system

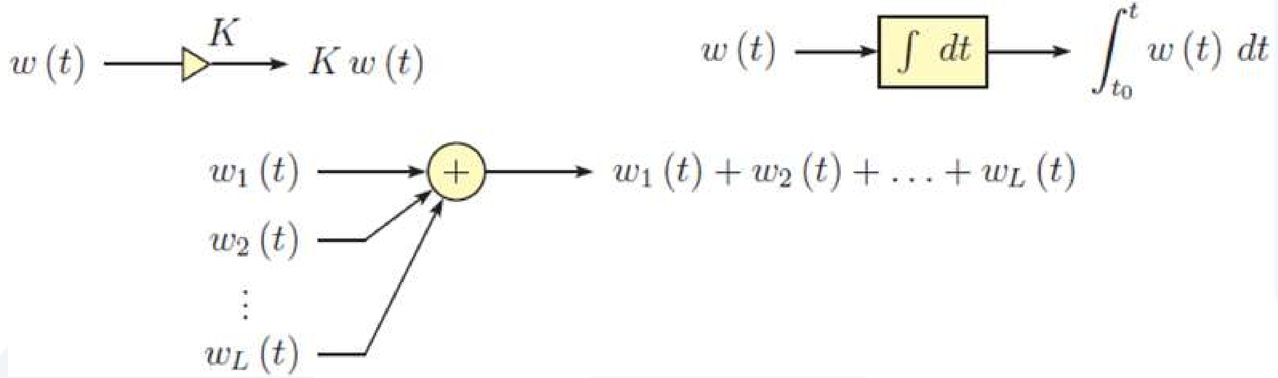






4. Block Diagram Representation of Continuous-Time Systems

- Block diagrams for CT systems are constructed using three types of components, namely constant-gain amplifiers, signal adders and integrators.

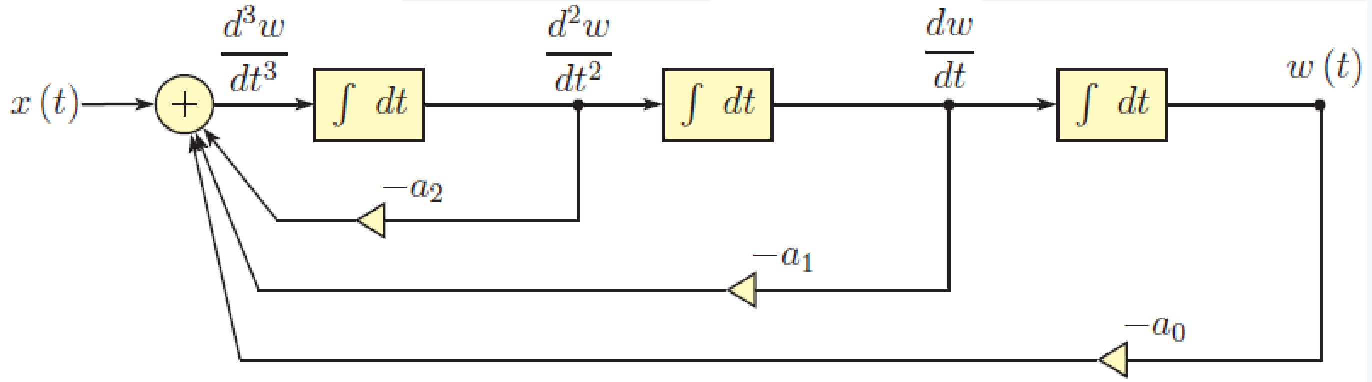


- The technique for finding a block diagram from a differential equation is best explained with an example.

$$\frac{d^3y}{dt^3} + a_2\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0y = b_2\frac{d^2x}{dt^2} + b_1\frac{dx}{dt} + b_0x$$

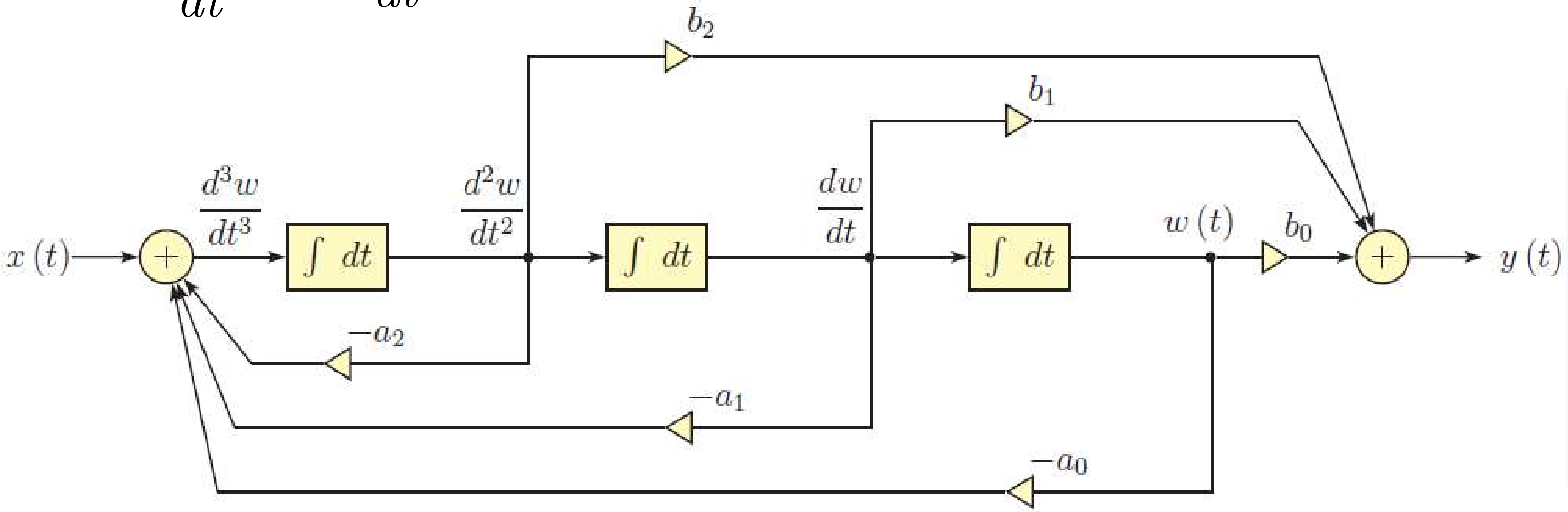
- we will introduce an intermediate variable $w(t)$

$$\frac{d^3w}{dt^3} + a_2\frac{d^2w}{dt^2} + a_1\frac{dw}{dt} + a_0w = x \Rightarrow \frac{d^3w}{dt^3} = x - a_2\frac{d^2w}{dt^2} - a_1\frac{dw}{dt} - a_0w$$



- The output signal $y(t)$ can be expressed in terms of the intermediate variable $w(t)$ as:

$$y = b_2 \frac{d^2w}{dt^2} + b_1 \frac{dw}{dt} + b_0 w$$

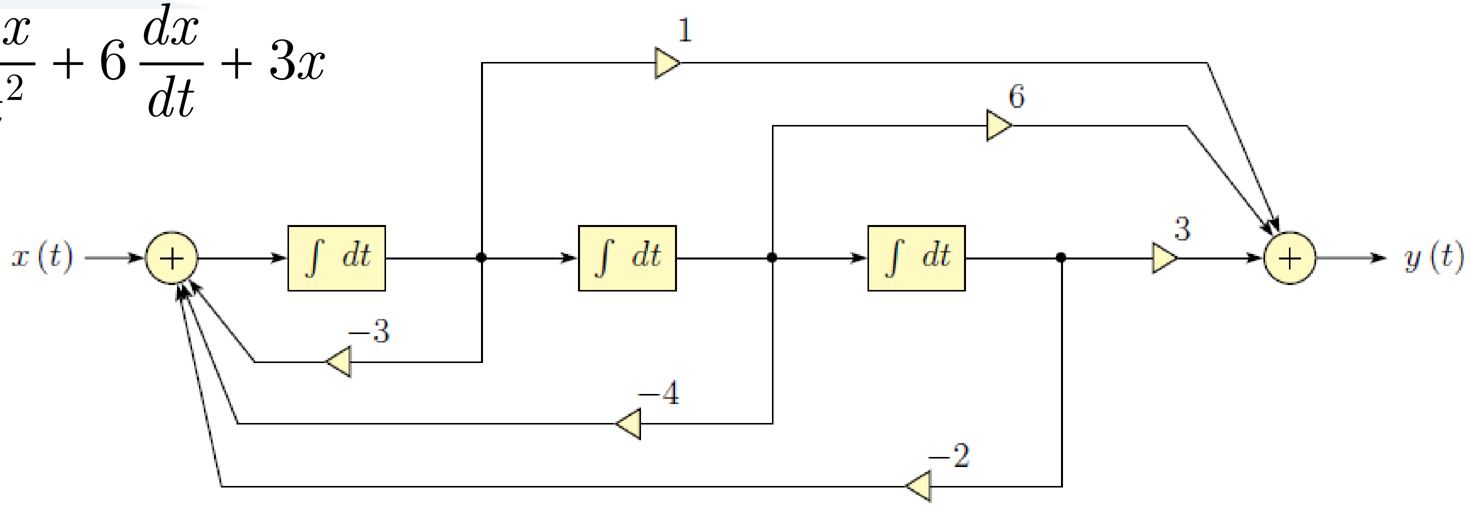


Imposing initial conditions

- Initial values of $y(t)$ and its first $N-1$ derivatives need to be converted to corresponding initial values of $w(t)$ and its first $N-1$ derivatives.

- Example 10: Block diagram for continuous-time system

$$\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 2y = \frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 3x$$



5. Impulse Response and Convolution

Convolution operation for CTLTI systems

- The (CT) convolution of the functions x and h, denoted $x * h$, is defined as the function:

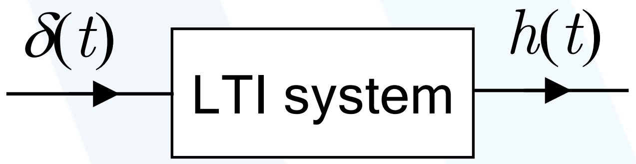
$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Properties of Convolution

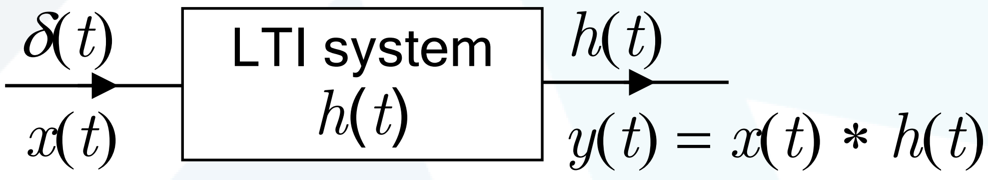
- Is commutative. For any two functions x and h, $x * h = h * x$.
- Is associative. For any functions x, h_1, and h_2, $(x * h_1) * h_2 = x * (h_1 * h_2)$.
- Is distributive with respect to addition. For any functions x, h_1, and h_2, $x * (h_1 + h_2) = x * h_1 + x * h_2$.
- For any function x, $x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau = x(t)$
- Moreover, δ is the convolutional identity. That is, for any function x, $x * \delta = x$.

Impulse response of a CTLTI system

- The response h of a system T to the input δ is called the impulse response of the system (i.e., $h = T\delta$).



- For any LTI system with input x, output y, and impulse response h, the following relationship holds: $y = x * h$.
- LTI system is completely characterized by its impulse response.
- That is, if the impulse response of a LTI system is known, we can determine the response of the system to any input.



Step Response of a CTLTI system

- The response $s(t)$ of a system T to the input $u(t)$ is called the step response of the system.

$$s(t) = \int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau = \int_{0}^{\infty} h(t-\tau)d\tau$$

- The impulse response h and step response s of a LTI system are related as:

$$h(t) = ds(t)/dt$$

- Example 11: Impulse response of the simple RC circuit

Consider the RC circuit. Let the element values be $R = 1\ \Omega$ and $C = 1/4$ F. Assume the initial value of the output at time $t = 0$ is $y(0) = 0$. Determine the impulse response of the system.

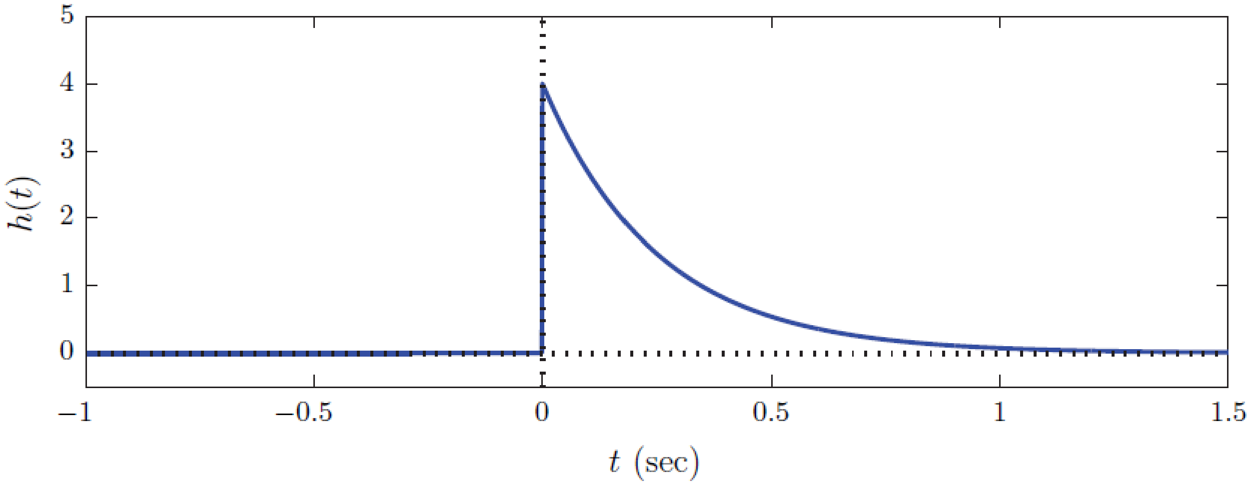
First method: using differential equation

$$y(t) = \int_0^t e^{-(t-\tau)/RC} \frac{1}{RC} x(\tau) d\tau$$

Setting $x(t) = \delta(t)$ $\quad h(t) = \int_0^t e^{-(t-\tau)/RC} \frac{1}{RC} \delta(\tau) d\tau = \frac{1}{RC} e^{-t/RC} u(t)$

Second method: unit-step response of the system

$$s(t) = (1 - e^{-t/RC}) u(t) \Rightarrow h(t) = \frac{ds(t)}{dt} = \frac{1}{RC} e^{-t/RC} u(t) = 4e^{-4t} u(t)$$



- Example 12: Impulse response of a second-order system (RLC circuit)

  Determine the impulse response of the RLC circuit that was used in Example 4. Use $R = 2\ \Omega$, $L = 1$ H and $C = 1/26$ F.

  First: find the unit-step response

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 26y(t) = 26x(t)$$

$$y_h(t) = c_1e^{-t}\cos(5t) + c_2e^{-t}\sin(5t), \qquad y_p(t) = 1$$

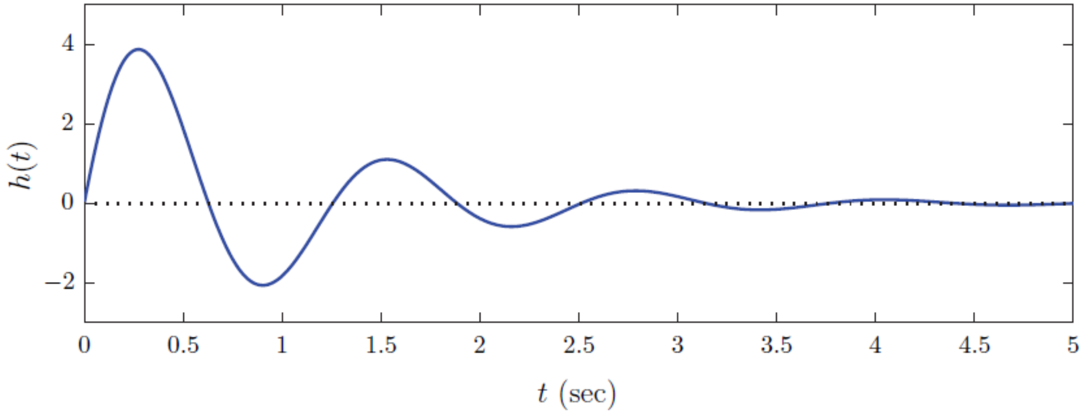
$$y(t) = y_h(t) + y_p(t) = c_1e^{-t}\cos(5t) + c_2e^{-t}\sin(5t) + 1$$

Assume that the system is CTLTI, and is therefore initially relaxed.

$$y(0) = 0 = c_1 + 1 \Rightarrow c_1 = -1, \quad \frac{dy}{dt}(0) = 0 = -c_1 + 5c_2 \Rightarrow c_2 = -0.2$$

$$s(t) = -e^{-t}\cos(5t) - 0.2e^{-t}\sin(5t) + 1, \quad t \geq 0$$

$$h(t) = \frac{ds(t)}{dt} = 5.2e^{-t}\sin(5t)u(t)$$



6. Causality and Stability in Continuous-Time Systems

- A system T is said to be causal if, for every real constant t_0, $T\{x(t_0)\}$ does not depend on $x(t)$ for some $t > t_0$.
- Acausal system is such that the value of its output at any given point in time can depend on the value of its input at only the same or earlier points in time.

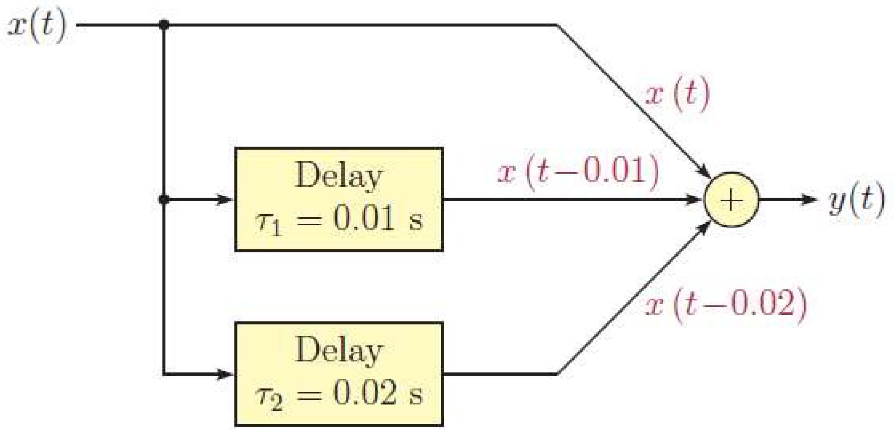
- For CTLTI systems the causality property can be related to the impulse response of the system $h(t) = 0$ for all $t < 0$.

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{0}^{\infty} h(\tau)x(t-\tau)d\tau$$

- Example 13: causal and non causal systems

  a. CT time-delay system $y(t) = x(t) + x(t - 0.01) + x(t - 0.02)$ √

  b. CT time-forward system $y(t) = x(t) + x(t + 0.1)$ X



- Note: A system must be causal in order to be physically realizable.
- A system is said to be stable in the bounded-input bounded-output sense if any bounded input signal is guaranteed to produce a bounded output signal.
- An input signal $x(t)$ is said to be bounded if an upper bound B_x exists such that $x(t) < B_x < \infty$ for all values of t.
- For stability of a continuous-time system: $x(t) < B_x < \infty \Rightarrow y(t) < B_y < \infty$
- For a CTLTI system to be stable, its impulse response must be absolute integrable.

$$\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$$

- Example 14: Stability of a first-order continuous-time system

  Evaluate the stability of the first-order CTLTI system described by the DE:

$$\frac{dy(t)}{dt} + ay(t) = x(t)$$

The step response of the system is when $x(t) = u(t)$

$$\frac{dy(t)}{dt} + ay(t) = u(t) \Rightarrow y(t) = ce^{-at} + \frac{1}{a}$$

$y(0) = 0$. (We take the initial value to be zero since the system is specified to be CTLTI. Non-zero initial conditions cannot be linear: Based on a zero input signal must produce a zero output signal).

$$y(0) = 0 \Rightarrow 0 = c + 1/a \Rightarrow c = -1/a$$

$$s(t) = \frac{1}{a}(1 - e^{-at})u(t) \qquad h(t) = \frac{ds(t)}{dt} = s(t) = e^{-at}u(t)$$

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{0}^{\infty} e^{-at}\, dt = \frac{1}{a}$$ Thus the system is stable if $a > 0$.